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Additive mappings between Hermitian matrix spaces preserving rank not exceeding one

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Abstract

Let K be a field of characteristic not two or three with an automorphism $-$ of order two. Let $F = \{a \in K : \bar{a} = a\}$. We characterize additive mappings T from one F-vector space of Hermitian matrices over K to another that preserve rank less than or equal to one for the following cases: (i) the image of T contains a matrix of rank at least three, (ii) every nonzero endomorphism of F is surjective.

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1. Introduction

In a previous paper [10], we describe those additive mappings from a second symmetric product space to another, over a field of characteristic not 2 or 3, which preserve decomposable elements of the form $\lambda u \cdot u$ where u is a vector and λ is a scalar. This leads to the corresponding result concerning additive mappings from one vector space of symmetric matrices to another which preserve rank less than or equal to one. Surjective additive mappings on symmetric matrices over a field of

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characteristic not 2 or 3 that preserve rank one were studied by Cao and Zhang [3]. In this note, we use the method in [10] and a nonbijective version of the fundamental theorem of projective geometry to obtain the structure of additive mappings T from one space of Hermitian matrices to another, over a field of characteristic not two or three, which preserve rank less than or equal to one under the assumption that the image of T contains a matrix of rank at least three. This generalizes a recent result of Tang [12] who characterizes additive mappings from one real vector space of $m \times m$ complex Hermitian matrices to another that preserve rank one under the hypothesis that the image of the identity matrix I_m has rank equal to m. The structure of surjective rank-one preserving additive mappings on Hermitian matrices over a division ring was obtained in [7] by using the fundamental theorem of geometry of Hermitian matrices. Characterizations of linear mappings from one real vector space of complex Hermitian matrices to another that preserve rank not exceeding one were obtained in [1,9]. Classifications of linear mappings on symmetric matrices over an infinite field of characteristic not two that preserve rank not exceeding an arbitrary fixed positive integer k were obtained by Loewy [11], while for fixed $k = 1, 2$, these were studied in [4,8]. The structure of linear mappings on symmetric matrices over an algebraically closed field of characteristic zero that preserve an arbitrary fixed rank was obtained by Beasley and Loewy [2].

2. Results

Throughout this paper, K denotes a field of characteristic not two or three with an automorphism $-$ of order two. Let $F = \{a \in K : \bar{a} = a\}$. Then K is a quadratic extension of F and there exists an element $i \in K \backslash F$ such that $\bar{i} = -i$ and $K = F(i)$. Let $\theta = i\bar{i}$. Let J denote the prime subfield of F. For each $m \times n$ matrix $A = (a_{ij})$ over K, let $A^* = (\bar{a}_{ij})^{\mathrm{t}}$.

Let K^m be the vector space of all $m \times 1$ column vectors over K. Let $H(m)$ be the F-vector space of all $m \times m$ Hermitian matrices over K. For any two vectors u, v of K^m, define

$$u \cdot v = \tfrac{1}{2}(uv^* + vu^*).$$

Then we have $u \cdot v = v \cdot u$, $(u_1 + u_2) \cdot v = u_1 \cdot v + u_2 \cdot v$ and for $c \in K$, $u \cdot cv = (\bar{c}u) \cdot v$. Let u^2 denote $u \cdot u$. Then a matrix A in $H(m)$ is of rank $k > 0$ if and only if $A = \sum_{i=1}^{k} \lambda_i u_i^2$ for some nonzero scalar λ_i in F, $i = 1, \ldots, k$ and some linearly independent vectors $u_1, \ldots, u_k$ in K^m. Let W be a nonzero subspace of K^m. Define $W^{(2)}$ to be the subspace of $H(m)$ generated by the vectors $u \cdot v$ where $u, v \in W$. It is easily checked that if $w_1, \ldots, w_k$ is a basis of W, then

$$\{w_s \cdot w_t : 1 \leqslant s \leqslant t \leqslant k\} \cup \{w_s \cdot iw_t : 1 \leqslant s < t \leqslant k\}$$

is a basis of $W^{(2)}$.

Let W be a subspace of K^m and V a subspace of K^n. Let $\sigma : K \to K$ be a nonzero endomorphism. A mapping $f : W \to V$ is called σ-quasilinear if $f(\lambda w) =$

$\sigma(\lambda)f(w)$ for all $\lambda \in K$ and $w \in W$. If in addition, σ is an automorphism, then f is called σ-semilinear. Let σ commute with the involution – and let $f : W \to V$ be a σ-quasilinear mapping. Then it induces a τ-quasilinear mapping $P(f)$ from $W^{(2)}$ to $V^{(2)}$ where $\tau = \sigma|_F$, such that

$$P(f)(u \cdot v) = f(u) \cdot f(v)$$

for all u, v in W. Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be the standard bases of K^m and K^n respectively. If $W = K^m$, $V = K^n$ and $Q = (q_{st})$ is the $n \times m$ matrix over K such that

$$f(e_i) = \sum_{j=1}^{n} q_{ji} f_j, \quad i = 1, \ldots, m.$$

Then

$$P(f)A = QA^{\sigma}Q^*$$

for all $A = (a_{ij})$ in $H(m)$ where $A^{\sigma} = (\sigma(a_{ij}))$.

A mapping ψ from $H(m)$ to $H(n)$ is called *rank-one nonincreasing* if $\rho(\psi(A)) \leqslant 1$ whenever $\rho(A) = 1$. Here ρ denotes the rank function. For any s vectors $w_1, \ldots, w_s$ in K^m, we use $\langle w_1, \ldots, w_s \rangle$ to denote the subspace spanned by $w_1, \ldots, w_s$.

For the following three lemmas, T denotes a rank-one nonincreasing additive mapping from $H(m)$ to $H(n)$.

Lemma 2.1. *Suppose that $T(au^2) \neq 0$, $a \in F$ and $u \in K^m$. Then there exists a scalar $c \in F$ such that $T(c^2u^2) \neq 0$. Moreover, $T(au^2)$ and $T(c^2u^2)$ are linearly dependent.*

Proof. It is same as that of Lemma 2.1 in [10]. □

Lemma 2.2. *Suppose that W is two-dimensional subspace of K^m such that $\dim\langle T(W^{(2)})\rangle \geqslant 2$. Then $T|_{W^{(2)}} = \lambda P(f)$ for some nonzero scalar $\lambda \in F$ and some σ-quasilinear mapping f from W to K^n where σ commutes with the automorphism $-$.*

Proof. Since $\dim\langle T(W^{(2)})\rangle \geqslant 2$, there exists two rank-one Hermitian matrices $c_1u_1^2$, $c_2u_2^2 \in W^{(2)}$ such that

$$T(c_iu_i^2) = \lambda_i v_i^2, \quad i = 1, 2$$

for some nonzero scalars $\lambda_1, \lambda_2 \in F$ and some linearly independent vectors $v_1, v_2 \in K^n$. By Lemma 2.1, we may assume that $c_1 = c_2 = 1$. Clearly, u_1 and u_2 are linearly independent, otherwise $\rho(u_1^2 + u_2^2) \leqslant 1$, but $T(u_1^2 + u_2^2)$ has rank 2, a contradiction. Let $S = \lambda_1^{-1}T$ and $\beta = \lambda_1^{-1}\lambda_2$. Note that

$$S(2u_1 \cdot u_2) = S(u_1 + u_2)^2 - v_1^2 - \beta v_2^2$$

is of rank $\leqslant 2$ and hence $S(u_1 + u_2)^2 \in \langle v_1, v_2\rangle^{(2)}$. This shows that

$$S(u_1 \cdot u_2) = av_1^2 + v_1 \cdot bv_2 + cv_2^2$$

for some $a, c \in F$ and $b \in K$. For any λ in the prime subfield J of F, we see that

$$S(u_1 + \lambda u_2)^2 = (1 + 2\lambda a)v_1^2 + 2\lambda v_1 \cdot bv_2 + (\beta\lambda^2 + 2\lambda c)v_2^2$$

and it is of rank less than 2, it follows that

$$\begin{vmatrix} 1 + 2\lambda a & \lambda\bar{b} \\ \lambda b & \lambda^2\beta + 2\lambda c \end{vmatrix} = 0.$$

As $\beta = \lambda_1^{-1}\lambda_2 \neq 0$, this implies that $a = c = 0$ and $\beta = b\bar{b}$. Hence $S(u_1 \cdot u_2) = v_1 \cdot bv_2$. Let $z_1 = v_1$ and $z_2 = bv_2$. Then we have

$$S(u_1^2) = z_1^2, \quad S(u_2^2) = z_2^2 \quad \text{and} \quad S(u_1 \cdot u_2) = z_1 \cdot z_2.$$

For any $\alpha \in F$, we have

$$S(\alpha u_1^2) = \sigma_1(\alpha)z_1^2 \quad \text{and} \quad S(\alpha u_2^2) = \sigma_2(\alpha)z_2^2,$$

where σ_1 and σ_2 are additive mappings on F with $\sigma_1(1) = \sigma_2(1) = 1$. Suppose that $S(\alpha u_1 \cdot u_2)$ is of rank one. Then $S(\alpha u_1 \cdot u_2 + u_1^2 + u_2^2)$ is of rank $\leqslant 2$ implies that $S(\alpha u_1 \cdot u_2) \in \langle z_1, z_2\rangle^{(2)}$. Suppose on the other hand that $S(\alpha u_1 \cdot u_2)$ is of rank 2. Then $S(\alpha u_1 \cdot u_2) = c_1w_1^2 + c_2w_2^2$ for some nonzero scalars $c_1, c_2 \in F$ and some linearly independent vectors $w_1, w_2 \in K^n$. Since $S(\alpha u_1 \cdot u_2 + u_i^2)$ is of rank $\leqslant 2$, it follows that $z_i \in \langle w_1, w_2\rangle$, $i = 1, 2$. Hence $\langle z_1, z_2\rangle = \langle w_1, w_2\rangle$. Thus

$$S(\alpha u_1 \cdot u_2) = \phi_1(\alpha)z_1^2 + \phi(\alpha)z_1 \cdot z_2 + \eta(\alpha)z_1 \cdot iz_2 + \phi_2(\alpha)z_2^2,$$

where $\phi_1, \phi, \eta, \phi_2$ are additive mappings on F such that

$$\phi(1) = 1, \quad \phi_1(1) = \phi_2(1) = \eta(1) = 0.$$

For any $\lambda, \delta \in J$ and $x, y \in F$, $S(\lambda xu_1 + \delta yu_2)^2$ is of rank $\leqslant 1$ and hence

$$\varDelta := \begin{vmatrix} \lambda^2\sigma_1(x^2) + 2\lambda\delta\phi_1(xy) & \lambda\delta(\phi(xy) - i\eta(xy)) \\ \lambda\delta(\phi(xy) + i\eta(xy)) & \delta^2\sigma_2(y^2) + 2\lambda\delta\phi_2(xy) \end{vmatrix} = 0.$$

Expanding the determinant and by considering the coefficient of $\lambda^3\delta$, we get $\sigma_1(x^2)\phi_2(xy) = 0$. Since $\sigma_1(1) = 1$, we obtain $\phi_2(y) = 0$ for any $y \in F$. Similarly, by considering the coefficient of $\delta^3\lambda$, we get $\phi_1(x) = 0$ for any $x \in F$. Hence

$$\varDelta = \lambda^2\delta^2(\sigma_1(x^2)\sigma_2(y^2) - \phi(xy)^2 - \theta\eta(xy)^2) = 0 \tag{1}$$

for any $\lambda, \delta \in J$ and $x, y \in F$. Putting $y = 1$ in (1), we get

$$\sigma_1(x^2) = \phi(x)^2 + \theta\eta(x)^2 \quad \text{for any } x \in F. \tag{2}$$

Putting $x = 1$ in (1) we get $\sigma_2(y^2) = \phi(y)^2 + \theta\eta(y)^2$ for any $y \in F$. Hence $\sigma_1(x^2) = \sigma_2(x^2)$. Since $\sigma_1((1 + x)^2) = (\phi(1 + x))^2 + \theta(\eta(1 + x))^2$ and σ_1, ϕ, and η are additive, it follows from (2) that $\sigma_1(x) = \phi(x)$ for any $x \in F$. Similarly, we have $\sigma_2 = \phi$. It follows from (1) that for any $x \in F$,

$$\sigma_1(x^2)\sigma_2(x^2) - \phi(x^2)^2 - \theta\eta(x^2)^2 = 0.$$

Hence $\eta(x^2) = 0$ since $\sigma_1 = \sigma_2 = \phi$. Since $\eta((1+x)^2) = 0$ and $\eta(1) = 0$, we get $\eta(x) = 0$. Hence $S(\alpha u_1 \cdot u_2) = \sigma(\alpha)z_1 \cdot z_2$. Also it follows from (2) that $\sigma_1(x^2) = \sigma_1(x)^2$. This together with the fact that σ_1 is additive imply that σ_1 is multiplicative. Hence σ_1 is a nonzero endomorphism on F. Let $\sigma = \sigma_1$. Since $\rho(S(u_1 + \lambda i u_2)^2) \leqslant 1$ for any $\lambda \in J$, it follows from arguments similar to the first paragraph of the proof that

$$S(u_1 \cdot iu_2) = z_1 \cdot hz_2 \quad \text{with } h\bar{h} = \sigma(\theta).$$

Since $S(u_1 + (1+i)u_2)^2 = z_1^2 + 2z_1 \cdot (1+h)z_2 + (1+\sigma(\theta))z_2^2$, and it is of rank $\leqslant 1$, it follows that $h + \bar{h} = 0$ and hence $h = ki$ for some $k \in F$. This proves that

$$S(u_1 \cdot iu_2) = z_1 \cdot kiz_2 \quad \text{and} \quad \sigma(\theta) = k^2\theta.$$

Using arguments similar to the beginning part of the second paragraph, we have for any $\alpha \in F$,

$$S(\alpha u_1 \cdot iu_2) = \psi_1(\alpha)z_1^2 + \psi(\alpha)z_1 \cdot z_2 + \varepsilon(\alpha)z_1 \cdot kiz_2 + \psi_2(\alpha)z_2^2,$$

where $\psi_1, \psi, \varepsilon, \psi_2$ are additive mappings on F such that

$$\varepsilon(1) = 1, \quad \psi_1(1) = \psi(1) = \psi_2(1) = 0.$$

For any $\lambda, \delta \in J$ and any $x \in F$, $S(\lambda x u_1 + \delta i u_2)^2$ is of rank $\leqslant 1$ and hence

$$\begin{vmatrix} \lambda^2\sigma(x^2) + 2\lambda\delta\psi_1(x) & \lambda\delta(\psi(x) - ik\varepsilon(x)) \\ \lambda\delta(\psi(x) + ik\varepsilon(x)) & \delta^2\sigma(\theta) + 2\lambda\delta\psi_2(x) \end{vmatrix} = 0.$$

This shows that $\psi_1(x) = 0$, $\psi_2(x) = 0$ for any $x \in F$ and hence

$$\sigma(x^2)\sigma(\theta) = \psi(x)^2 + k^2\theta\varepsilon(x)^2$$

for any $x \in F$. Since $\sigma(\theta) = k^2\theta$, we have

$$\sigma(x^2)\sigma(\theta) = \psi(x)^2 + \sigma(\theta)\varepsilon(x)^2. \tag{3}$$

Replacing x by $1 + x$ in (3) we get

$$\sigma((1+x)^2)\sigma(\theta) = (\psi(1+x))^2 + \sigma(\theta)(\varepsilon(1+x))^2. \tag{4}$$

It follows from (3) and (4) that $\sigma(2x) = \varepsilon(2x)$ for any $x \in F$. Hence $\sigma = \varepsilon$ and it follows from (3) that $\psi = 0$. Thus

$$S(\alpha u_1 \cdot iu_2) = \sigma(\alpha)z_1 \cdot kiz_2$$

for any $\alpha \in F$. Extend σ to an endomorphism on K by defining $\sigma(i) = ki$, we see that $S|_{W^{(2)}} = P(f)$ where f is the σ-quasilinear mapping from W to K^n such that $f(u_1) = z_1$, $f(u_2) = z_2$. Clearly σ commutes with the automorphism $-$. $\square$

It is easily checked that f in Lemma 2.2 sends every two linearly independent vectors to two linearly independent vectors.

Lemma 2.3. *If $T(x_1^2) = cy^2 \neq 0$ and $T(x_2^2) = dy^2$ for some $c, d \in F$ and some $y \in K^n$, then $T(x_1 \cdot sx_2) \in \langle y^2 \rangle$ for any $s \in K$.*

Proof. Suppose the contrary. Then $T(\langle x_1, x_2\rangle^{(2)})$ spans a linear subspace of dimension $\geqslant 2$ and hence by Lemma 2.2, $T(x_1^2)$ and $T(x_2^2)$ are linearly independent, a contradiction. □

For the following two lemmas we assume that $T : H(m) \to H(n)$ is a rank-one nonincreasing additive mapping such that $\dim\langle T|_{W^{(2)}}\rangle \geqslant 2$ for some two-dimensional subspace W of K^m generated by u_1, u_2.

Lemma 2.4. *If $T(z^2) = 0$ where $z \in K^m\backslash\{0\}$, then*

$$T(z \cdot h) = 0 \quad \textit{for any } h \in K^m.$$

Proof. Suppose the contrary that $T(z \cdot h) \neq 0$ for some $h \in K^m - \{0\}$. Then either $T(z + h)^2 \neq 0$ or $T(z - h)^2 \neq 0$. Without loss of generality, we may assume that $T(z_1^2) \neq 0$ where $z_1 = z + h$. Choose $z_2 \in \langle u_1, u_2\rangle$ such that $T(z_2^2)$ and $T(z_1^2)$ are linearly independent. By Lemma 2.2, we have $T(z_i^2) = cw_i^2, \quad i = 1, 2$ and $T(z_1 \cdot z_2) = cw_1 \cdot w_2$ for some linearly independent vectors w_1, w_2 in K^n and some nonzero scalar $c \in F$. In view of Lemma 2.3,

$$T(z_1 \cdot z) = aw_1^2, \quad T(z_2 \cdot z) = bw_2^2$$

for some scalars a and b in F. Note that for any $\lambda \in J$,

$$T(z + \lambda z_1 + z_2)^2 = \lambda^2 cw_1^2 + cw_2^2 + 2\lambda cw_1 \cdot w_2 + 2\lambda aw_1^2 + 2bw_2^2$$

is of rank $\leqslant 1$ and hence

$$(\lambda^2 c + 2\lambda a)(c + 2b) - \lambda^2 c^2 = 0.$$

This implies that $b = a = 0$ and $T(z_1 \cdot z) = 0$. Since $T(z^2) = 0$, it follows that $T(z \cdot h) = 0$, a contradiction, and the proof is complete. □

Lemma 2.5. *Let $Z = \{x \in K^m : T(x^2) = 0\}$. Then Z is a subspace of K^m.*

Proof. Let $u, v \in Z - \{0\}$ and $\lambda \in K$. Then $T(\lambda u)^2 = T(u \cdot \lambda\bar{\lambda}u) = 0$ by Lemma 2.4. We have $T(u + v)^2 = T(u)^2 + 2T(u \cdot v) + T(v^2) = 0$ by Lemma 2.4. Hence Z is a subspace of K^m.

Theorem 2.6. *Let $T : H(m) \to H(n)$ be a rank-one nonincreasing additive mapping such that* Im T *contains a matrix of rank at least* 3. *Then there exist a $n \times m$ matrix Q, a nonzero endomorphism σ of K commuting with the automorphism $-$ and a nonzero scalar λ in F such that*

$$T(A) = \lambda QA^\sigma Q^* \quad \textit{for all } A \in H(m).$$

Proof. Let $Z = \{x \in K^m : T(x^2) = 0\}$. Then Z is a subspace of K^m. Let Y be a complementary subspace of Z. By Lemma 2.4, $\operatorname{Im} T = \operatorname{Im} T|_{Y^{(2)}}$. Since T is rank-one nonincreasing and $\operatorname{Im} T$ contains a matrix of rank $\geqslant 3$, it follows that

$T(x_1^2), T(x_2^2), T(x_3^2)$ are linearly independent for some vectors x_1, x_2, x_3 in Y. Clearly x_1, x_2 are linearly independent. Let $W = \langle x_1, x_2\rangle$. Then by Lemma 2.2, $T|_{W^{(2)}} = \lambda P(f)$ for some $\lambda \in F\backslash\{0\}$ and some σ-quasilinear mapping from W to K^n. Moreover, σ commutes with the automorphism $-$. Let $\psi = \lambda^{-1}T$. Let $P(Y)$ denote the projective space of Y consisting of all one-dimensional subspaces of Y. Define a mapping θ from $P(Y)$ to $P(K^n)$ as follows:

$$\phi(\langle x\rangle) = \langle v\rangle \quad \text{if } \psi(x^2) \in \langle v^2\rangle.$$

Clearly ϕ is a well-defined mapping. If $\langle x\rangle \subseteq \langle y\rangle + \langle z\rangle$ where $x, y, z \in Y\backslash\{0\}$, Lemmas 2.2 or 2.3 imply that $\phi(\langle x\rangle) \subseteq \phi(\langle y\rangle) + \phi(\langle z\rangle)$ and hence ϕ is a morphism between the projective spaces $P(Y)$ and $P(K^n)$. Since the image of ϕ is not contained in a line of $P(K^n)$, by the nonbijective version of the fundamental theorem of projective geometry [6, Theorem 10.1.3], there exists a quasilinear mapping $g : Y \to K^n$ such that

$$\phi(\langle x\rangle) = \langle g(x)\rangle$$

for any $\langle x\rangle \in P(Y)$. Since $\langle g(x)\rangle = \langle f(x)\rangle$ for any $x \in W - \{0\}$, it follows that $f|_W = dg|_W$ for some scalar d in K. Now for any $y \in Y\backslash W$, choose $x \in W$ such that $g(x)$ and $g(y)$ are linearly independent. We have $\psi(y^2) = c_y g(y)^2$ for some $c_y \in F$. By Lemma 2.2, we see that $\psi(x \cdot y) = g(x) \cdot cg(y)$ for some $c \in K$. Hence

$$\begin{aligned}\psi((x+y)^2) &= c_{x+y}g(x+y)^2 \quad \text{for some } c_{x+y} \in F\\ &= d\bar{d}g(x)^2 + 2g(x)\cdot cg(y) + c_y g(y)^2.\end{aligned}$$

This shows that $c_{x+y} = d\bar{d} = c_y$. Hence

$$\psi|_{Y^{(2)}} = P(dg).$$

Extend g to be the σ-quasilinear mapping such that $g(z) = 0$ for all $z \in Z$. In view of Lemma 2.4, we have $\psi = P(dg)$. Therefore $T = \lambda P(dg)$ and the proof is complete. □

Theorem 2.7. *Let $T : H(m) \to H(n)$ be a rank-one nonincreasing additive mapping. Suppose that every nonzero endomorphism of F is surjective. Then one of the following is true*:

(i) *$T(A) = \phi(A)v^2$ for some additive functional ϕ from $H(m)$ to F and some nonzero vector $v \in K^n$,*

(ii) *there exist an $n \times m$ matrix Q, an automorphism σ on K commuting with $-$ and a nonzero scalar λ in F such that*

$$T(A) = \lambda QA^\sigma Q^* \quad \textit{for all } A \in H(m).$$

Proof. *Case 1.* Im $T \subseteq \langle v^2\rangle$ for some nonzero vector $v \in K^m$. For each $A \in H(m)$, $T(A) = \phi(A)v^2$ for some $\phi(A) \in F$. Clearly ϕ is an additive mapping.

Case 2. There exist $u_1, u_2 \in K^m$ such that $T(u_1^2)$ and $T(u_2^2)$ are linearly independent. Let $W = \langle u_1, u_2\rangle$. Then by Lemma 2.2, $T|_{W^{(2)}} = \lambda P(f)$ for some nonzero scalar λ in F and some injective semilinear mapping $f : W \to K^n$ associated with an automorphism σ on K where σ commutes with the automorphism $-$. Let $\psi = \lambda^{-1}T$. Let Y be a complimentary subspace of $Z := \{x \in K^m : \psi(x^2) = 0\}$ such that $Y \supseteq W$. Suppose that $\dim Y = 2$. If g is the σ-semilinear mapping from K^m to K^n such that $g|_W = f$ and $\ker g = Z$, then we see from Lemma 2.4 that $\psi = P(g)$. Now we assume that $\dim Y \geqslant 3$. Let $x \in Y\backslash W$. Then $\psi(x^2) = cu^2$ for some $c \in F\backslash\{0\}$ and some $u \in K^n\backslash\{0\}$. We shall show that $u \notin \operatorname{Im} f$. Suppose the contrary. Let $z \in W$ such that $f(z)$ and u are linearly independent. Let $f(z) = v$. From the proof of Lemma 2.2, we see that there exists $w = bu$, $b \in K$ such that

$$\psi(x^2) = w^2, \quad \psi(z \cdot x) = w \cdot v.$$

Note that $\langle u\rangle = \langle w\rangle$ and hence $w \in \operatorname{Im} f$. Let $y \in W$ such that $f(y) = w$. Then $\psi(z \cdot y) = w \cdot v$. In view of Lemma 2.3, $\psi(x \cdot y) = aw^2$ for some $a \in F$. Since

$$\psi((x + y + z)^2) = (2 + 2a)w^2 + 4w \cdot v + v^2$$

is of rank $\leqslant 1$, it follows that $a = 1$. Thus, we have $\psi((x - y)^2) = 0$, a contradiction. This shows that $u \notin \operatorname{Im} f$ and hence $\operatorname{Im} T$ contains a matrix of rank 3. The result now follows from Theorem 2.6.

Remark 2.8. Let T be a semilinear rank-one nonincreasing mapping from $H(m)$ to $H(n)$. Assume that $|F| > 3$ (instead of char$F \neq 3$). From the proof of Theorem 2.7 we see that either (i) $T(A) = \phi(A)v^2$ for some semilinear functional ϕ from $H(m)$ to F and some nonzero vector $v \in K^n$ or (ii) there exist an $n \times m$ matrix Q, an automorphism σ on K commuting with $-$, and a nonzero scalar λ in F such that $T(A) = \lambda QA^\sigma Q^*$ for all A in $H(m)$.

Remark 2.9. Let $T : H(m) \to H(n)$ be a nonzero additive mapping ($n \geqslant m \geqslant 2$) such that $\rho(T(A) + T(B)) = \rho(T(A)) + \rho(T(B)) \leqslant m$ whenever $\rho(A + B) = \rho(A) + \rho(B)$. Using Lemma 2.2 together with Theorem 2.6, and similar arguments (with slight modification) as in the proof of [5, Theorem 3.3], we can show that there exist an $n \times m$ matrix Q of rank m, a nonzero endomorphism σ on K commuting with –, and a nonzero scalar λ in F such that $T(A) = \lambda QA^\sigma Q^*$ for all A in $H(m)$. When $H(m)$ is the real vector space of $m \times m$ complex Hermitian matrices, this result was proved by Tang in [12] under the restriction that $m = n$ and T is injective.

Let H be a complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators on H. For any vectors x and y in H, let $x \otimes y$ be the linear operator on H defined by $(x \otimes y)z = (z, y)x (z \in H)$ and let $x \cdot y = \frac{1}{2}(x \otimes y + y \otimes x)$. Let $F_s(H)$ denote the real vector space of all finite rank self-adjoint operators in $B(H)$. Using the same arguments as in the proof of Theorem 2.7, we have the following result.

Theorem 2.10. *Let H and M be two complex Hilbert spaces. Let $T : F_s(H) \to F_s(M)$ be an additive mapping such that* $\operatorname{rank}(T(A)) \leqslant 1$ *whenever* $\operatorname{rank}(A) = 1$. *Then one of the following is true:*

(i) $T(A) = \phi(A)(v \otimes v)$ *for some additive functional ϕ from $F_s(H)$ to the real field and some nonzero vector v in M;*
(ii) $T(x \otimes x) = \lambda\psi(x) \otimes \psi(x)$ *for all $x \in H$, for some nonzero real number λ and some linear or conjugate linear operator ψ from H to M.*

Remark 2.11. The corresponding result of Theorem 2.10 for real Hilbert spaces H and M is also true where ψ in (ii) is now a linear operator. This can be shown by modifying the proof of Theorem 2.7.

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